Kuwait University — Math 101 — Date: May 5, 2008
Dept. of Math. & Comp. Sci. — Second Exam — Duration: 90 minutes

**Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed**

Answer the following questions:

1. (a) Use differentials to approximate $\sqrt[3]{8.12}$. (2 pts.)

   (b) Find $\frac{dy}{dx}$, where $y = \sec\left(\frac{\sqrt{x}}{x^2+1}\right)$. (2 pts.)

2. Show that $f(x) = (x+1)^{\frac{5}{4}} + x^{\frac{1}{3}}$, has a vertical tangent line. (3 pts.)

3. Find an equation of the tangent line to the curve $3y^3 + 4xy - x^2 \sin y = 3$ at the point $P(0,1)$.

   (3 pts.)

4. (a) State The Mean Value Theorem.

   (b) Let $f$ and $g$ be two differentiable functions on $[a,b]$ satisfying: $f(a) = g(a)$, $f'(x) < g'(x)$, for all $x$ in $(a,b)$. Use the Mean Value Theorem to show that

   $f(b) < g(b)$. [Hint: use $h(x) = g(x) - f(x)$]

   (4 pts.)

5. Two points $A$ and $B$ located at the origin $(0,0)$ of the $xy-$plane. Point $A$ moves along the positive $x-$axis at $3m/\min$, and point $B$ moves along the positive $y-$axis at $4m/\min$. What is the rate of change of the distance between $A$ and $B$ after two minutes? (3 pts.)

6. Let $f(x) = \frac{x}{1-x^2}$ and given that $f'(x) = \frac{1+x^2}{(1-x^2)^2}$ and $f''(x) = \frac{2x(x^2+3)}{(1-x^2)^3}$.

   (a) Find the vertical and horizontal asymptotes for the graph of $f$, if any.

   (b) Find the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing. Find the local extrema of $f$, if any.

   (c) Find the intervals on which the graph of $f$ is concave upward and the intervals on which the graph of $f$ is concave downward. Find the points of inflection, if any.

   (d) Sketch the graph of $f$. (8 pts.)

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1. (a) Let $f(x) = \sqrt[3]{x}$ and $x_0 = 8$, then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and , $\Delta x = 0.12$. So, $\sqrt[3]{8.12} = f(8.12) \simeq f(8) + f'(8)\Delta x = 2 + \frac{1}{12}(0.12) = \boxed{2.01}$.

   (b) $\dfrac{dy}{dx} = \left[\dfrac{\frac{1}{2\sqrt{x}}(x^2+1) - 2x\sqrt{x}}{(x^2+1)^2}\right] \sec\left(\dfrac{\sqrt{x}}{x^2+1}\right)\tan\left(\dfrac{\sqrt{x}}{x^2+1}\right)$
   $= \left[\dfrac{1-3x^2}{2\sqrt{x}(x^2+1)^2}\right]\sec\left(\dfrac{\sqrt{x}}{x^2+1}\right)\tan\left(\dfrac{\sqrt{x}}{x^2+1}\right)$.

2. $\boxed{f'(x) = \frac{5}{4}(x+1)^{\frac{1}{4}} + \frac{1}{3x^{2/3}}}$. Since $f$ is ***continuous*** at $x = 0$ and $\lim_{x\to 0^{\pm}} f'(x) = \infty$, then $f$ has a vertical tangent at $x = 0$.

3. Differentiate implicitly with respect to $x$, we have:

   $9y^2y' + 4y + 4xy' - 2x\sin y - x^2y'\cos y = 0$. Therefore, $\boxed{y'|_{(0,1)} = -\frac{4}{9}}$.

   Equation of tangent line: $\boxed{4x + 9y - 9 = 0}$.

4. (b) Let $h(x) = g(x) - f(x)$. So, $h(a) = 0$ $(g(a) = f(a))$. Since $f$ and $g$ are differentiable on $[a, b]$, then

   (1) $f$ and $g$ and hence $h$ are continuous on $[a, b]$

   (2) $f$ and $g$ and hence $h$ are differentiable on $(a, b)$, and $h'(x) = g'(x) - f'(x) > 0$ for all $x \in (a, b)$ $(f'(x) < g'(x)$ for all $x \in (a, b))$.

   From The Mean Value Theorem $\exists\ c \in (a, b)$ such that $h(b) - h(a) = h'(c)(b - a)$. Since, $b > a, h'(c) > 0$ and $h(a) = 0$, then $h(b) > 0 \implies g(b) - f(b) > 0 \implies f(b) < g(b)$.

5. Let the coordinates of $A$ and $B$, after time $t$, be $(x, 0)$ and $(0, y)$, respectively and $S$ be the distance between $A$ and $B$. Thus, $S = \sqrt{x^2 + y^2}$.

   Solution (1): Differentiate with respect to $t$, $\dfrac{dS}{dt} = \dfrac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$. After two minutes,

   $x = 3(2) = 6m/\min$ and $y = 4(2) = 8m/\min$. Thus, $\boxed{\left.\dfrac{dS}{dt}\right|_{t=2} = 5m/\min.}$

   Solution (2): $S = \sqrt{x^2+y^2}$, $x = 3t$, $y = 4t \implies S = \sqrt{(3t)^2 + (4t)^2} = 5t \implies$ $\boxed{\dfrac{dS}{dt} = 5m/\min}$.

6. Domain $f = \mathbb{R} - \{-1, 1\}$. The point $(0, 0)$ lies on the curve. $f(-x) = -f(x)$, then $f$ is an odd function. The graph of $f$ is symmetric about the origin.

   (a) $\lim_{x\to -1^{\pm}} f(x) = \mp\infty$ and $\lim_{x\to 1^{\pm}} f(x) = \mp\infty \implies \boxed{x = -1}$ and $\boxed{x = 1}$ are Vertical Asymptotes. $\lim_{x\to\pm\infty} f(x) = 0 \implies \boxed{y = 0}$ is a Horizontal Asymptote.

   (b) $f'(x) \neq 0$. At $x = -1$ and $x = 1$, $f'$does not exist ($f$ has *infinite discontinuity*).

| I | $(-\infty, -1)$ | $(-1, 1)$ | $(1, \infty)$ |
|---|---|---|---|
| sign of $f'(x)$ | + | + | + |
| Conclusion | $\nearrow$ | $\nearrow$ | $\nearrow$ |

   $f$ is increasing on $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. $f$ has no local extrema.

(c) $f''(0) = 0$, and $f''$ does not exist at $x = \pm 1$ where $f$ is not continuous.

| I | $(-\infty, -1)$ | $(-1, 0)$ | $(0, 1)$ | $(1, \infty)$ |
|---|---|---|---|---|
| sign of $f''(x)$ | $+$ | $-$ | $+$ | $-$ |
| Concavity | $CU$ | $CD$ | $CU$ | $CD$ |

, $(0, 0)$ is an inflection point.

(d)